A gauge-invariant discrete analog of the Yang-Mills equations on a double complex

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Abstract

An intrinsically defined gauge-invariant discrete model of the Yang-Mills equations on a combinatorial analog of $\mathbb{R}^4$ is constructed. We develop several algebraic structures on the matrix-valued cochains (discrete forms) that are analogs of objects in differential geometry. We define a combinatorial Hodge star operator based on the use of a double complex construction. Difference self-dual and anti-self-dual equations will be given. In the last section we discus the question of generalizing our constructions to the case of a 4-dimensional combinatorial sphere.

Key words and phrases: Yang-Mills equations, gauge invariance, difference equations

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1 Introduction

In this paper we construct a gauge-invariant discrete model of the Yang-Mills equations and study combinatorial analogs of some objects in differential geometry, namely the Hodge star operator, the self-dual and anti-self-dual equations. We define these structures on a combinatorial analog of $\mathbb{R}^4$ based on the use of a double complex.

Using the approach first introduced by Dezin [7], in [19, 20] we consider gauge-invariant discrete models of the Yang-Mills equations in $\mathbb{R}^n$ and in Minkowski space. Our approach based also on some constructions from [8], where certain 2-dimensional models connected with the Yang-Mills equations are discussed. In [19] the combinatorial Hodge star operator $*$ is defined using both an inner product on discrete matrix-valued forms (cochains) and Poincaré duality but the operation $(*)^2$ is equivalent to a shift with corresponding sign. This

is one of the main distinctive features of the formalism [7] as compared to the continual case, where the operator $(*)^2$ is either an involution or antiinvolution.

In with paper we introduce a combinatorial object, namely a double complex, in which the discrete Hodge star operator is defined in such way that $(*)^2 = \pm Id$. At the same time we consider discrete forms, the product $\cup$ on cochains (discrete analog of the exterior product) and the coboundary operator d^c (discrete analog of the exterior differential operator) similarly as in [19, 20].

There is another approach presented in Dodziuk's paper [9]. In [9, 10] the authors using an embedding of simplicial cochains into differential forms (due to [22]) show that a combinatorial Laplacian on the cochains provide a good approximation of the smooth Laplacian on a closed Riemannian manifold. Using the techniques [9], Wilson [23] defines a combinatorial star operator on the simplicial cochains of a triangulated Riemannian manifold and proves its convergence to the smooth star operator. Other related results on the subject can be found in [3, 13, 4].

In section 4 we construct a discrete analog of the Yang-Mills equations on the cochains of the double complex. We try to be as close to continual Yang-Mills theory as possible. We'll define the discrete Yang-Mills equations using both a geometric structure of the object and a gauge invariance of these equations.

A large number of paper in the physical literature have been devoted to discretization of gauge theories (see, for example, [2, 1, 17, 5, 16] and the references therein). Discrete version of Yang-Mills theories using lattices and graphs, as well as their applications to finite dimensional versions of gauge theories, have been studied in [15, 18, 12]. Some other interesting results on gauge invariant lattice models of Yang-Mills actions using the geometry of finite groups can be found in [6].

It is well known that in 4-dimensional non-abelian gauge theory the self-dual and anti-self-dual connections are the most important extrema of the Yang-Mills action. In section 5 we study discrete analogs of the self-dual and anti-self-dual equations on the double complex and show that some interesting relations amongst the curvature form and its self-dual and anti-self-dual parts, that hold in the continual theory, also hold in the combinatorial case. We also describe difference self-dual equations as a system of nonlinear matrix equations.

In section 6 we consider a combinatorial construction which is homeomorphic to a 4-dimensional sphere. The technique introduced, namely the combinatorial Hodge star operator on the cochains of the double complex, allows us to describe a discrete model of the Yang-Mills equations on a combinatorial 4-dimensional sphere. Using the same approach a discrete analog of the Laplacian on a combinatorial 2-dimensional sphere is considered in [21]. It is interesting to study problems like this since the question concerning the global approximations throughout the surface of a ball has not been study enough.

2 Continual setting

In this section we briefly recall some well known definitions of smooth Yang-Mills theory (see, for example, [14]). Let M be a smooth oriented Riemannian manifold. Consider the trivial bundle $P = M \times SU(2)$. Let T^*P be the cotangent bundle of P and let (x, g), $x \in M$, $g \in SU(2)$, be local coordinates of the bundle P. It is known [14] that a connection can be shown to arise from a certain 1-form $\omega \in T^*P$, where ω is required to have values in the Lie algebra $su(2)$ of the Lie group $SU(2)$. This form is given by

$$\omega = g^{-1}dg + g^{-1}Ag. \tag{2.1}$$

The connection 1-form A is defined by

$$A = \sum_{\alpha,\mu} A^{\alpha}_{\mu}(x)\lambda_{\alpha}dx^{\mu}, \tag{2.2}$$

where λ_α is a basis of $su(2)$ and $A^{\alpha}_{\mu}(x)$ is a smooth function for any μ, α. Here we take $\lambda_\alpha = \frac{\sigma_\alpha}{2i}$, where $\sigma_\alpha,\ \alpha = 1, 2, 3$, are the standard Pauli matrices and i is the imaginary unit.

Let the coordinates of P change (locally) from (x, g) to (x', g'). Let us only make a change of fibre coordinates, i.e. $x' = x$ and g' is given by

$$g' = hg, \qquad h \in SU(2). \tag{2.3}$$

Under the change of coordinates (2.3) the 1-form ω induces a certain transformation law for the connection form A. Suppose that the form (2.1) is invariant under transformations (2.3), i.e.

$$(g')^{-1}dg' + (g')^{-1}A'g' = g^{-1}dg + g^{-1}Ag.$$

From with we obtain

$$A' = hdh^{-1} + hAh^{-1}. \tag{2.4}$$

This is the transformation law of the connection form A. It is what is called in Yang-Mills theories the gauge transformation law.

We define the curvature 2-form F in the following way

$$F = dA + A \wedge A. \tag{2.5}$$

Under the gauge transformation (2.3) the curvature F changes as follows

$$F' = hFh^{-1}. \tag{2.6}$$

Define the covariant exterior differential operator d_A by

$$d_A\Omega = d\Omega + A \wedge \Omega + (-1)^{r+1}\Omega \wedge A, \tag{2.7}$$

where Ω is a $su(2)$-valued r-form.

Then the Yang-Mills equations can be written as

$$d_A F = 0, \tag{2.8}$$

$$d_A * F = 0, \tag{2.9}$$

where $*$ is the Hodge star operator. Equation (2.8) is known as the Bianchi identity. Let Φ, Ψ be $su(2)$-valued r-forms on M. The inner product is defined by

$$(\Phi, \Psi) = -tr \int_M \Phi \wedge *\Psi, \tag{2.10}$$

where tr is the trace operator.

Let M be 4-dimensional. The Yang-Mills action S can be expressed in terms of the 2-forms F and $*F$ as

$$S = -tr \int_M F \wedge *F. \tag{2.11}$$

Equations (2.8), (2.9) are the Euler-Lagrange equations for the extrema of S. In 4-dimensional Yang-Mills theories the following equations

$$F = *F, \qquad F = - * F \tag{2.12}$$

are called self-dual and anti-self-dual respectively. Solutions of (2.12) – the self-dual and anti-self-dual connections – are called also instantons and antiinstantons [11]. It is known that the self-dual and anti-self-dual connections are the most important minima of the action S.

3 Double complex

Let the tensor product $C(4) = C \otimes C \otimes C \otimes C$ of an 1-dimensional complex C be a combinatorial model of Euclidean space $\mathbb{R}^4$ (see for details [7, 20]). The 1-dimensional complex C is defined in the following way. Let C^0 denotes the real linear space of 0-dimensional chains generated by basis elements x_κ (points), $\kappa \in \mathbb{Z}$. It is convenient to introduce the shift operators τ, σ in the set of indices by

$$\tau\kappa = \kappa + 1, \qquad \sigma\kappa = \kappa - 1.$$

We denote the open interval $(x_\kappa,\ x_{\tau\kappa})$ by e_κ. We'll regards the set $\{e_\kappa\}$ as a set of basis elements of the real linear space C^1 of 1-dimensional chains. Then the 1-dimensional complex (combinatorial real line) is the direct sum of the introduced spaces $C = C^0 \oplus C^1$. The boundary operator ∂ on the basis elements of C is given by

$$\partial x_\kappa = 0, \qquad \partial e_\kappa = x_{\tau\kappa} - x_\kappa. \tag{3.1}$$

The definition is extended to arbitrary chains by linearity.

Multiplying the basis elements x_κ, e_κ in various way we obtain basis elements of $C(4)$. Let $s_k^{(p)}$, where $k = (k_1, k_2, k_3, k_4)$, $k_i \in \mathbb{Z}$, be an arbitrary basis element of $C(4)$. We suppose that the superscript (p) contains the whole requisite

information about the number and places of the 1-dimensional "components" e_κ in $s_k^{(p)}$. For example, 1-dimensional basis elements of $C(4)$ can be written as

$$e_k^1 = e_{k_1} \otimes x_{k_2} \otimes x_{k_3} \otimes x_{k_4}, \qquad e_k^2 = x_{k_1} \otimes e_{k_2} \otimes x_{k_3} \otimes x_{k_4},$$
$$e_k^3 = x_{k_1} \otimes x_{k_2} \otimes e_{k_3} \otimes x_{k_4}, \qquad e_k^4 = x_{k_1} \otimes x_{k_2} \otimes x_{k_3} \otimes e_{k_4} \tag{3.2}$$

and for the 2-dimensional basis elements ε_k^{ij} we have

$$\varepsilon_k^{12} = e_{k_1} \otimes e_{k_2} \otimes x_{k_3} \otimes e_{k_4}, \qquad \varepsilon_k^{23} = x_{k_1} \otimes e_{k_2} \otimes e_{k_3} \otimes x_{k_4},$$
$$\varepsilon_k^{13} = e_{k_1} \otimes x_{k_2} \otimes e_{k_3} \otimes x_{k_4}, \qquad \varepsilon_k^{24} = x_{k_1} \otimes e_{k_2} \otimes x_{k_3} \otimes e_{k_4},$$
$$\varepsilon_k^{14} = e_{k_1} \otimes x_{k_2} \otimes x_{k_3} \otimes e_{k_4}, \qquad \varepsilon_k^{34} = x_{k_1} \otimes x_{k_2} \otimes e_{k_3} \otimes e_{k_4}. \tag{3.3}$$

Using (3.1), we define the boundary operator ∂ on chains of $C(4)$ in the following way: if $c_p,\ c_q$ are chains of the indicated dimension, belonging to the complexes being multiplied, then

$$\partial(c_p \otimes c_q) = \partial c_p \otimes c_q + (-1)^p c_p \otimes \partial c_q. \tag{3.4}$$

For example, for the basis element ε_k^{24} we have

$$\begin{aligned}\partial\varepsilon_k^{24} &= \partial(x_{k_1} \otimes e_{k_2}) \otimes x_{k_3} \otimes e_{k_4} - x_{k_1} \otimes e_{k_2} \otimes \partial(x_{k_3} \otimes e_{k_4})\\ &= \partial x_{k_1} \otimes e_{k_2} \otimes x_{k_3} \otimes e_{k_4} + x_{k_1} \otimes \partial e_{k_2} \otimes x_{k_3} \otimes e_{k_4}\\ &- x_{k_1} \otimes e_{k_2} \otimes \partial x_{k_3} \otimes e_{k_4} - x_{k_1} \otimes e_{k_2} \otimes x_{k_3} \otimes \partial e_{k_4}\\ &= x_{k_1} \otimes x_{\tau k_2} \otimes x_{k_3} \otimes e_{k_4} - x_{k_1} \otimes x_{k_2} \otimes x_{k_3} \otimes e_{k_4}\\ &- x_{k_1} \otimes x_{k_2} \otimes x_{k_3} \otimes x_{\tau k_4} + x_{k_1} \otimes x_{k_2} \otimes x_{k_3} \otimes x_{k_4}.\end{aligned}$$

We now describe the construction of a double complex. Together with the complex $C(4)$ we consider its "double", namely the complex $\tilde{C}(4)$ of exactly the same structure. Define the one-to-one correspondence

$$* : C(4) \to \tilde{C}(4), \qquad * : \tilde{C}(4) \to C(4) \tag{3.5}$$

in the following way. Let $s_k^{(p)}$ be an arbitrary p-dimensional basis element of $C(4)$, i.e. the product $s_k^{(p)} = s_{k_1} \otimes s_{k_2} \otimes s_{k_3} \otimes s_{k_4}$ contains exactly p 1-dimensional elements e_{k_i} and $4-p$ 0-dimensional elements x_{k_i}, $p = 0,1,2,3,4$, $k_i \in \mathbb{Z}$. Then

$$* : s_k^{(p)} \to \pm\tilde{s}_k^{(4-p)}, \qquad * : \tilde{s}_k^{(4-p)} \to \pm s_k^{(p)}, \tag{3.6}$$

where

$$\tilde{s}_k^{(4-p)} = *s_{k_1} \otimes *s_{k_2} \otimes *s_{k_3} \otimes *s_{k_4}$$

and $*s_{k_i} = \tilde{e}_{k_i}$ if $s_{k_i} = x_{k_i}$ and $*s_{k_i} = \tilde{x}_{k_i}$ if $s_{k_i} = e_{k_i}$. In the first of mapping (3.6) we take "+" if the permutation $((p),\ (4-p))$ of $(1,2,3,4)$ is even and "$-$" if the permutation $((p),\ (4-p))$ is odd. Recall that in symbol (p) the number of components is contained. For example, for the 2-dimensional basis element $\varepsilon_k^{13} = e_{k_1} \otimes x_{k_2} \otimes e_{k_3} \otimes x_{k_4}$ we have $*\varepsilon_k^{13} = -\tilde{\varepsilon}_k^{24}$ since the permutation $(1,3,2,4)$ is odd. The mapping $* : \tilde{s}_k^{(4-p)} \to \pm s_k^{(p)}$ is defined by analogy.

Proposition 3.1. *Let $c_r \in C(4)$ be an r-dimensional chain. Then we have*

$$* * c_r = (-1)^{r(4-r)} c_r. \tag{3.7}$$

Proof. For $r = 0, 4$ it is obviously. Let $r = 1$, then for the 1-dimensional basis elements (3.2) we have

$$*e_k^1 = \tilde{x}_{k_1} \otimes \tilde{e}_{k_2} \otimes \tilde{e}_{k_3} \otimes \tilde{e}_{k_4} = \tilde{e}_k^{234}, \qquad *e_k^2 = -\tilde{e}_{k_1} \otimes \tilde{x}_{k_2} \otimes \tilde{e}_{k_3} \otimes \tilde{e}_{k_4} = -\tilde{e}_k^{134},$$
$$*e_k^3 = \tilde{e}_{k_1} \otimes \tilde{e}_{k_2} \otimes \tilde{x}_{k_3} \otimes \tilde{e}_{k_4} = \tilde{e}_k^{124}, \qquad *e_k^4 = -\tilde{e}_{k_1} \otimes \tilde{e}_{k_2} \otimes \tilde{e}_{k_3} \otimes \tilde{x}_{k_4} = -\tilde{e}_k^{123}$$

and

$$*\tilde{e}_k^{123} = e_k^4, \quad *\tilde{e}_k^{124} = -e_k^3, \quad *\tilde{e}_k^{134} = e_k^2, \quad *\tilde{e}_k^{234} = -e_k^1.$$

Hence $* * e_k^i = -e_k^i$ for any $i = 1, 2, 3, 4$. The case $r = 3$ is similar.

Let now $\varepsilon_k^{ij} \in C(4)$ be a 2-dimensional basis element (3.3) . Then

$$* * \varepsilon_k^{12} = *\tilde{\varepsilon}_k^{34} = \varepsilon_k^{12}, \quad * * \varepsilon_k^{13} = - * \tilde{\varepsilon}_k^{24} = \varepsilon_k^{13}, \quad * * \varepsilon_k^{14} = *\tilde{\varepsilon}_k^{23} = \varepsilon_k^{14},$$
$$* * \varepsilon_k^{23} = *\tilde{\varepsilon}_k^{14} = \varepsilon_k^{23}, \quad * * \varepsilon_k^{24} = - * \tilde{\varepsilon}_k^{13} = \varepsilon_k^{24}, \quad * * \varepsilon_k^{34} = *\tilde{\varepsilon}_k^{12} = \varepsilon_k^{34}.$$

To an arbitrary chain c_r the operation $*$ extends by linearity. □

Now we consider a dual object of the complex $C(4)$. Let $K(4)$ be a cochain complex with $gl(2,\mathbb{C})$-valued coefficients, where $gl(2,\mathbb{C})$ is the Lie algebra of all complex 2×2 matrices. We suppose that the complex $K(4)$, which is a conjugate of $C(4)$, has a similar structure: $K(4) = K \otimes K \otimes K \otimes K$, where K is a dual of the 1-dimensional complex C. Basis elements of K can be written as $\{x^\kappa\}, \{e^\kappa\}$. Then an arbitrary basis element of $K(4)$ is given by $s^k = s^{k_1} \otimes s^{k_2} \otimes s^{k_3} \otimes s^{k_4}$, where s^{k_j} is either x^{k_j} or e^{k_j}. For example, we denote the 1-, 2-dimensional basis elements of $K(4)$ by e_i^k, ε_{ij}^k respectively, cf. (3.2), (3.3).

We define the pairing operation for arbitrary basis elements $\varepsilon_k \in C(4)$, $s^k \in K(4)$ by the rule

$$< \varepsilon_k, as^k >= \begin{cases} 0, & \varepsilon_k \neq s_k \\ a, & \varepsilon_k = s_k, \ a \in gl(2,\mathbb{C}). \end{cases} \tag{3.8}$$

The operation (3.8) is linearly extended to cochains. We will call cochains forms, emphasizing their relationship with the corresponding continual objects, differential forms.

The operation ∂ (3.4) induces the dual operation d^c on $K(4)$ in the following way:

$$< \partial\varepsilon_k, as^k >=< \varepsilon_k, ad^c s^k > . \tag{3.9}$$

The coboundary operator d^c is an analog of the exterior differentiation operator.

Now we describe a cochain product on the forms of $K(4)$. See [7, 19, 20] for details. We denote this product by $\cup$. In terms of the homology theory this is the so-called Whitney product. First we introduce the $\cup$-product on the chains

of the 1-dimensional complex K. For the basis elements of K the $\cup$-product is defined as follows

$$x^\kappa \cup x^\kappa = x^\kappa, \quad e^\kappa \cup x^{\tau\kappa} = e^\kappa, \quad x^\kappa \cup e^\kappa = e^\kappa, \quad \kappa \in \mathbb{Z},$$

supposing the product to be zero in all other case. To arbitrary forms the $\cup$-product be extended linearly. Let us introduce an r-dimensional complex $K(r)$, $r = 1, 2, 3$, in an obvious notation. Let $s^k_{(p)}$ be an arbitrary p-dimensional basis element of $K(r)$. It is convenient to write the basis element of $K(r+1)$ in the form $s^k_{(p)} \otimes s^\kappa$, where $s^k_{(p)}$ is a basis element of $K(r)$ and s^κ is either e^κ or x^κ, $\kappa \in \mathbb{Z}$. Then, supposing that the $\cup$-product in $K(r)$ has been defined, we introduce it for basis elements of $K(r+1)$ by the rule

$$(s^k_{(p)} \otimes s^\kappa) \cup (s^k_{(q)} \otimes s^\mu) = Q(\kappa, q)(s^k_{(p)} \cup s^k_{(q)}) \otimes (s^\kappa \cup s^\mu), \tag{3.10}$$

where the signum function $Q(\kappa, q)$ is equal to -1 if the dimension of both elements s^κ, $s^k_{(q)}$ is odd and to $+1$ otherwise (see [7]). The extension of the $\cup$-product to arbitrary forms of $K(r+1)$ is linear. Note that the coefficients of forms multiply as matrices.

Proposition 3.2. *Let φ and ψ be arbitrary forms of $K(4)$. Then*

$$d^c(\varphi \cup \psi) = d^c\varphi \cup \psi + (-1)^p \varphi \cup d^c\psi, \tag{3.11}$$

where p is the dimension of a form φ.

The proof of Proposition 3.2 is totally analogous to one in [7, p. 147] for the case of discrete forms with real coefficients.

The complex of the cochains $\tilde{K}(4)$ over the double complex $\tilde{C}(4)$, with the operator d^c defined in it by (3.9), has the same structure as $K(4)$. The operation (3.9) induces the respective mapping

$$* : K(4) \to \tilde{K}(4), \qquad * : \tilde{K}(4) \to K(4)$$

by the rule:

$$< \tilde{c},\ *\varphi > = < *\tilde{c},\ \varphi >, \qquad < c,\ *\tilde{\psi} > = < *c,\ \tilde{\psi} >, \tag{3.12}$$

where $c \in C(4)$, $\tilde{c} \in \tilde{C}(4)$, $\varphi \in K(4)$, $\tilde{\psi} \in \tilde{K}(4)$. It is obviously that Proposition 3.1 is true for any r-dimensional cochain $c^r \in K(4)$. So we have

$$** \varphi = (-1)^{r(4-r)}\varphi$$

for any discrete r-form φ on $K(4)$ and note that the same relation holds in the continual case.

Let $V \subset C(4)$ be a "domain" of the complex $C(4)$. We define its as follows

$$V = \sum_k V_k, \qquad k = (k_1, k_2, k_3, k_4), \quad k_i = 1, 2, ..., N_i, \tag{3.13}$$

where $V_k = e_{k_1} \otimes e_{k_2} \otimes e_{k_3} \otimes e_{k_4}$ is a 4-dimensional basis element of $C(4)$. Let $s_k^{(p)}$ be a p-dimensional basis element of $C(4)$. We set

$$V_p = \sum_k \sum_{(p)} s_k^{(p)} \otimes *s_k^{(p)}, \tag{3.14}$$

where the subscripts $k_i,\ i = 1, 2, 3, 4$, run the set of values indicated in (3.13). For example,

$$V_1 = \sum_k \sum_{i=1}^{4} e_k^i \otimes *e_k^i = \sum_k (e_k^1 \otimes \tilde{e}_k^{234} - e_k^2 \otimes \tilde{e}_k^{134} + e_k^3 \otimes \tilde{e}_k^{124} - e_k^4 \otimes \tilde{e}_k^{123}).$$

Let $K(V)$ denotes $K(4)$ restricted to V and let

$$\mathbb{V} = \sum_{p=0}^{4} V_p.$$

Consider the following discrete p-forms

$$\varphi = \sum_k \sum_{(p)} \varphi_k^{(p)} s^k_{(p)}, \qquad \varphi^* = \sum_k \sum_{(p)} \big(\varphi_k^{(p)}\big)^* s^k_{(p)},$$

where $\varphi_k^{(p)} \in gl(2, \mathbb{C})$ and $\big(\varphi_k^{(p)}\big)^*$ denotes the conjugate transpose of the matrix $\varphi_k^{(p)}$, i. e. $\big(\varphi_k^{(p)}\big)^* = \big(\overline{\varphi}_k^{(p)}\big)^T$.

For any p-forms $\varphi, \psi \in K(V)$ we define the inner product $(\ ,\ )_V$ by

$$\begin{aligned}(\varphi,\ \psi)_V &= tr < \mathbb{V},\ \varphi \otimes *\psi^* >= tr < V_p,\ \varphi \otimes *\psi^* > \\ &= tr \sum_k \sum_{(p)} < s_k^{(p)},\ \varphi >< *s_k^{(p)},\ *\psi^* > .\end{aligned} \tag{3.15}$$

Using (3.6), (3.8), it is easy to check that

$$(\varphi,\ \psi)_V = tr \sum_k \sum_{(p)} \varphi_k^{(p)} \big(\psi_k^{(p)}\big)^*, \tag{3.16}$$

where $\varphi_k^{(p)}, \big(\psi_k^{(p)}\big)^* \in gl(2, \mathbb{C})$ are components of $\varphi, \psi^* \in K(V)$.

Note that for $su(2)$-valued p-forms on V (cf. (2.10)) Relation (3.16) can be rewritten as follows

$$(\varphi,\ \psi)_V = -tr < \mathbb{V},\ \varphi \otimes *\psi >= -tr \sum_k \sum_{(p)} \varphi_k^{(p)} \psi_k^{(p)}.$$

The inner product makes it possible to define the adjoint of d^c, denoted δ^c.

Proposition 3.3. *For any $(p-1)$-form φ and p-form ω we have*

$$(d^c\varphi,\ \omega)_V = tr < \partial\mathbb{V},\ \varphi \otimes *\omega^* > +(\varphi,\ \delta^c\omega)_V, \tag{3.17}$$

where

$$\delta^c = (-1)^p *^{-1} d^c * \tag{3.18}$$

and $**^{-1} = Id$.

Proof. The proof is a computation. From the definition (3.9) it follows that (3.4) induces the similar relation for the coboundary operator d^c on forms:

$$d^c(\varphi \otimes *\omega) = d^c\varphi \otimes *\omega + (-1)^{p-1}\varphi \otimes d^c(*\omega).$$

Using this, we compute

$$\begin{aligned}(d^c\varphi,\ \omega)_V &= tr < \mathbb{V},\ d^c\varphi \otimes *\omega^* >= tr < V_p,\ d^c\varphi \otimes *\omega^* > \\ &= tr < \mathbb{V},\ d^c(\varphi \otimes *\omega^*) > -(-1)^{p-1} tr < V_{p-1},\ \varphi \otimes d^c(*\omega^*) > \\ &= tr < \partial\mathbb{V},\ \varphi \otimes *\omega^* > +(-1)^p tr < \mathbb{V},\ \varphi \otimes *(*^{-1} d^c * \omega)^* > .\end{aligned}$$

It immediately follows (3.17). □

Relation (3.17) is a discrete analog of the Green formula. It should be noted that from (3.7) we have:

$$*^{-1} = (-1)^{p(4-p)} * .$$

4 Discrete Yang-Mills equations

In this section we'll construct a discrete model of the Yang-Mills equations (2.8), (2.9) using the double complex introduced above. Let $A \in K(4)$ be a discrete 1-form. We can write A as

$$A = \sum_k \sum_{i=1}^4 A_k^i e_i^k, \tag{4.1}$$

where $A_k^i \in su(2)$ and e_i^k is an 1-dimensional basis element of $K(4)$, $k = (k_1, k_2, k_3, k_4)$, $k_i \in \mathbb{Z}$. Suppose that the $su(2)$-valued 1-form (4.1) is a discrete analog of the connection form (2.2).

Let us introduce some discrete 0-dimensional form with coefficients belonging to the Lie group $SU(2)$. We put

$$h = \sum_k h_k x^k, \tag{4.2}$$

where $h_k \in SU(2)$ and $x^k = x^{k_1} \otimes x^{k_2} \otimes x^{k_3} \otimes x^{k_4}$ is a 0-dimensional basis element of $K(4)$.

The discrete analog of the transformations (2.3), (2.4) are defined to be

$$g' = h \cup g, \qquad A' = h \cup d^c h^{-1} + h \cup A \cup h^{-1}, \tag{4.3}$$

where h, h^{-1}, g are 0-forms of the type (4.2) and h^{-1} denotes the form with inverse coefficients (inverse matrices). We'll call this transformation a gauge transformation for the discrete model.

Remark 4.1. *The set of the 0-forms (4.2) is a group with respect to the* $\cup$-*product.*

It is obviously, since by definition of the $\cup$-product for the 0-forms $h,\ g$ we have

$$h \cup g = \Big(\sum_k h_k x^k\Big) \cup \Big(\sum_k g_k x^k\Big) = \sum_k h_k g_k x^k,$$

where $h_k,\ g_k$ are multiplied as matrices.

The discrete curvature form F is defined by

$$F = d^c A + A \cup A. \tag{4.4}$$

The 2-form $F \in K(4)$ we can write also as follows

$$F = \sum_k \sum_{i<j} F_k^{ij} \varepsilon_{ij}^k, \tag{4.5}$$

where $F_k^{ij} \in gl(2,\mathbb{C})$, ε_{ij}^k is a 2-dimensional basis elements of $K(4)$ and $1 \le i,j \le 4$, $k = (k_1,k_2,k_3,k_4)$, $k_i \in \mathbb{Z}$.

Let us introduce for convenient the shifts operator τ_i and σ_i as

$$\tau_i k = (k_1, ...\tau k_i, ...k_4), \qquad \sigma_i k = (k_1, ...\sigma k_i, ...k_4).$$

Similarly, we denote by τ_{ij} (σ_{ij}) the operator shifting to the right (to the left) two differ components of $k = (k_1,k_2,k_3,k_4)$. For example,

$$\tau_{12} k = (\tau k_1, \tau k_2, k_3, k_4), \qquad \sigma_{14} k = (\sigma k_1, k_2, k_3, \sigma k_4).$$

Combining (4.4) and (4.5) and using (3.8) – (3.10), we obtain

$$F_k^{ij} = \Delta_{k_i} A_k^j - \Delta_{k_j} A_k^i + A_k^i A_{\tau_i k}^j - A_k^j A_{\tau_j k}^i, \tag{4.6}$$

where $\Delta_{k_i} A_k^j = A_{\tau_i k}^j - A_k^j$.

Remark 4.2. *In the continual case the curvature form F (2.5) takes values in the algebra $su(2)$ for any $su(2)$-valued connection form A. Unfortunately, it is not true in the discrete case because, generally speaking, the components $A_k^i A_{\tau_i k}^j - A_k^j A_{\tau_j k}^i$ of the form $A \cup A$ (see (4.6)) do not belong to $su(2)$.*

It is easy to check that the combinatorial Bianchi identity:

$$d^c F = A \cup F - F \cup A \tag{4.7}$$

holds for the discrete curvature form (4.4) (cf. (2.8)). The discrete analog of the exterior covariant differentiation operator (2.7) is defined by

$$d_A^c \Omega = d^c \Omega + A \cup \Omega + (-1)^{r+1} \Omega \cup A, \tag{4.8}$$

where Ω is an arbitrary r-form of $K(4)$.

Theorem 4.3. *Under the gauge transformation (4.3) the curvature form (4.4) changes as*

$$F' = h \cup F \cup h^{-1}. \tag{4.9}$$

Proof. The proof is similar to that of Proposition 2, [20]. $\square$

Let us introduce the following operation

$$\tilde{\iota} : K(4) \to \tilde{K}(4), \qquad \tilde{\iota} : \tilde{K}(4) \to K(4)$$

by setting

$$\tilde{\iota} s^k_{(p)} = \tilde{s}^k_{(p)}, \qquad \tilde{\iota} \tilde{s}^k_{(p)} = s^k_{(p)}, \tag{4.10}$$

where $s^k_{(p)}$ and $\tilde{s}^k_{(p)}$ are basis elements of $K(4)$ and $\tilde{K}(4)$. So, for a p-form $\varphi \in K(4)$ we have $\tilde{\iota}\varphi = \tilde{\varphi}$. Recall that the coefficients of $\tilde{\varphi} \in \tilde{K}(4)$ and $\varphi \in K(4)$ are the same.

Proposition 4.4. *The following hold*

$$\tilde{\iota}^2 = Id, \quad \tilde{\iota}* = *\tilde{\iota}, \quad \tilde{\iota} d^c = d^c \tilde{\iota}, \tag{4.11}$$
$$\tilde{\iota}(\varphi \cup \psi) = \tilde{\iota}\varphi \cup \tilde{\iota}\psi,$$

where $\varphi,\ \psi \in K(4)$.

Proof. The proof immediately follows from definitions of the corresponding operations. $\square$

The discrete analog of Equation (2.9) can be written as

$$d^c_A * \tilde{\iota} F = 0. \tag{4.12}$$

Using (4.8), we have

$$d^c_A * \tilde{\iota} F = d^c * \tilde{\iota} F + A \cup *\tilde{\iota} F - *\tilde{\iota} F \cup A. \tag{4.13}$$

Lemma 4.5. *Let h be a discrete 0-form. Then for an arbitrary p-form $f \in K(4)$ we have*

$$\tilde{\iota} * (h \cup f) = h \cup \tilde{\iota} * f. \tag{4.14}$$

Proof. Applying (4.10), the proof is analogous to the proof of Lemma 1 in [20]. $\square$

Lemma 4.6. *Let $f \in K(4)$ be a 2-form. We have*

$$\tilde{\iota} * (f \cup h) = \tilde{\iota} * f \cup h \tag{4.15}$$

if and only if the coefficients of a 0-form h satisfy the following conditions

$$h_{\tau_{12}k} = h_{\tau_{34}k}, \qquad h_{\tau_{13}k} = h_{\tau_{24}k}, \qquad h_{\tau_{14}k} = h_{\tau_{23}k} \tag{4.16}$$

for all $k = (k_1, k_2, k_3, k_4),\ k_i \in \mathbb{Z}$.

Proof. The proof is computational. See also the proof of Lemma 2 in [20]. Using (3.10) and (3.6), we compute

$$f \cup h = \sum_k \sum_{i<j} f_k^{ij} h_{\tau_{ij}k} \varepsilon_{ij}^k,$$

and

$$*f = \sum_k (f_k^{12} \tilde{\varepsilon}_{34}^k - f_k^{13} \tilde{\varepsilon}_{24}^k + f_k^{14} \tilde{\varepsilon}_{23}^k + f_k^{23} \tilde{\varepsilon}_{14}^k - f_k^{24} \tilde{\varepsilon}_{13}^k + f_k^{34} \tilde{\varepsilon}_{12}^k),$$

where ε_{ij}^k is a 2-dimensional basis element of $K(4)$. Then, by the definition of $\tilde{\iota}$, we obtain

$$\begin{aligned}\tilde{\iota} * (f \cup h) = \sum_k (& f_k^{12} h_{\tau_{12}k} \varepsilon_{34}^k - f_k^{13} h_{\tau_{13}k} \varepsilon_{24}^k + f_k^{14} h_{\tau_{14}k} \varepsilon_{23}^k \\ &+ f_k^{23} h_{\tau_{23}k} \varepsilon_{14}^k - f_k^{24} h_{\tau_{24}k} \varepsilon_{13}^k + f_k^{34} h_{\tau_{34}k} \varepsilon_{12}^k).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\tilde{\iota} * f \cup h = \sum_k (& f_k^{12} h_{\tau_{34}k} \varepsilon_{34}^k - f_k^{13} h_{\tau_{24}k} \varepsilon_{24}^k + f_k^{14} h_{\tau_{23}k} \varepsilon_{23}^k \\ &+ f_k^{23} h_{\tau_{14}k} \varepsilon_{14}^k - f_k^{24} h_{\tau_{13}k} \varepsilon_{13}^k + f_k^{34} h_{\tau_{12}k} \varepsilon_{12}^k).\end{aligned}$$

Combining the last two expressions with one another, we conclude that (4.15) implies (4.16) and vice versa. □

It should be noted that the set of 0-forms (4.2), which satisfy Conditions (4.16), is a group under $\cup$-product (see Remark 4.1).

Theorem 4.7. *Under Conditions (4.16) the discrete Yang-Mills equation (4.12) is gauge invariant.*

Proof. The proof is analogous to the proof of Theorem 1 in [20]. By Theorem 4.2 and Lemmas 4.4, 4.5, we have

$$\tilde{\iota} * F' = \tilde{\iota} * (h \cup F \cup h^{-1}) = h \cup \tilde{\iota} * F \cup h^{-1}.$$

Note that h^{-1} also satisfies Conditions (4.16). Using (3.18) we compute

$$d^c \tilde{\iota} * F' = d^c h \cup \tilde{\iota} * F \cup h^{-1} + h \cup d^c \tilde{\iota} * F \cup h^{-1} + h \cup \tilde{\iota} * F \cup d^c h^{-1}.$$

Since $d^c h \cup h^{-1} = -d^c h \cup h^{-1}$ and taking into account (4.3) and (4.9), we get

$$A' \cup \tilde{\iota} * F' = -d^c h \cup \tilde{\iota} * F \cup h^{-1} + h \cup A \cup \tilde{\iota} * F \cup h^{-1}$$

and

$$\tilde{\iota} * F' \cup A' = h \cup \tilde{\iota} * F \cup d^c h^{-1} + h \cup \tilde{\iota} * F \cup A \cup h^{-1}.$$

Putting the last three expressions in (4.13), one obtains:

$$\begin{aligned}d^c_{A'} \tilde{\iota} * F' &= h \cup d^c \tilde{\iota} * F \cup h^{-1} + h \cup A \cup \tilde{\iota} * F \cup h^{-1} \\ &- h \cup \tilde{\iota} * F \cup A \cup h^{-1} = h \cup d^c_A \tilde{\iota} * F \cup h^{-1}.\end{aligned}$$

Thus, if $d^c_A \tilde{\iota} * F = 0$, then $d^c_{A'} \tilde{\iota} * F' = 0$.

□

5 Difference self-dual and anti-self-dual equations

The discrete analog of Equations (2.12) is defined by

$$F = \tilde{\iota} * F, \qquad F = -\tilde{\iota} * F, \tag{5.1}$$

where F is the discrete curvature form (4.4). Using (4.5), by the definitions of $\tilde{\iota}$ and $*$, the first equation (self-dual) of (5.1) can be rewritten as follows

$$F_k^{12} = F_k^{34}, \qquad F_k^{13} = -F_k^{24}, \qquad F_k^{14} = F_k^{23}. \tag{5.2}$$

We call these equations difference self-dual equations. Using (4.6), we obtain

$$\Delta_{k_1} A_k^2 - \Delta_{k_2} A_k^1 + A_k^1 A_{\tau_1 k}^2 - A_k^2 A_{\tau_2 k}^1 = \Delta_{k_3} A_k^4 - \Delta_{k_4} A_k^3 + A_k^3 A_{\tau_3 k}^4 - A_k^4 A_{\tau_4 k}^3,$$

$$\Delta_{k_1} A_k^3 - \Delta_{k_3} A_k^1 + A_k^1 A_{\tau_1 k}^3 - A_k^3 A_{\tau_3 k}^1 = \Delta_{k_4} A_k^2 - \Delta_{k_2} A_k^4 - A_k^2 A_{\tau_2 k}^4 + A_k^4 A_{\tau_4 k}^2,$$

$$\Delta_{k_1} A_k^4 - \Delta_{k_4} A_k^1 + A_k^1 A_{\tau_1 k}^4 - A_k^4 A_{\tau_4 k}^1 = \Delta_{k_2} A_k^3 - \Delta_{k_3} A_k^2 + A_k^2 A_{\tau_2 k}^3 - A_k^3 A_{\tau_3 k}^2.$$

Recall that $A_k^i \in su(2)$ is a component of the discrete connection 1-form (4.1).

Obviously, changing the sign on the right hand side of Equations (5.2), we obtain the difference anti-self-dual equations.

As in the continual case (see, for example, [14]), we can decompose our arbitrary discrete 2-form F into its self-dual and anti-self-dual parts as follows

$$F = F^+ + F^-,$$

where $F^+ = \frac{1}{2}(F + \tilde{\iota} * F)$ and $F^- = \frac{1}{2}(F - \tilde{\iota} * F)$. The form F^+ is self-dual, i.e. $F^+ = \tilde{\iota} * F^+$. Indeed, using Proposition 3.1 and (4.11), we compute

$$\tilde{\iota} * F^+ = \tilde{\iota} * \frac{1}{2}(F + \tilde{\iota} * F) = \frac{1}{2}(\tilde{\iota} * F + \tilde{\iota}^2 *^2 F) = \frac{1}{2}(\tilde{\iota} * F + F) = F^+.$$

Similarly,

$$\tilde{\iota} * F^- = \tilde{\iota} * \frac{1}{2}(F - \tilde{\iota} * F) = \frac{1}{2}(\tilde{\iota} * F - \tilde{\iota}^2 *^2 F) = \frac{1}{2}(\tilde{\iota} * F - F) = -F^-.$$

So, F^- is anti-self-dual.

Let $\| \ \|_V$ denote the norm on $K(V)$ generated by the inner product (3.15). Then a discrete analog of the Yang-Mills action (2.11) can be written as

$$S = \|F\|_V^2 = (F,\ F)_V = tr < V_2,\ F \otimes *F^* > .$$

See also (3.13)–(3.16).

Theorem 5.1. *For the discrete curvature form (4.4) we have*

$$\|F\|_V^2 = \|F^+\|_V^2 + \|F^-\|_V^2. \tag{5.3}$$

Proof. By definition (3.15) we have

$$\begin{aligned}\|F\|^2_V &= \|F^+ + F^-\|^2_V \\ &= tr < V_2,\ F^+ \otimes *(F^+)^* > + tr < V_2,\ F^- \otimes *(F^-)^* > \\ &+ tr < V_2,\ F^+ \otimes *(F^-)^* > + tr < V_2,\ F^- \otimes *(F^+)^* > \\ &= \|F^+\|^2_V + \|F^-\|^2_V + (F^+,\ F^-)_V + (F^-,\ F^+)_V.\end{aligned}$$

Denote the components of $F^+,\ F^-$ by $(F^{ij}_k)^+,\ (F^{ij}_k)^-$ respectively. For $(F^{ij}_k)^+$ we have (5.2) and for $(F^{ij}_k)^-$ we can write the following relations

$$(F^{12}_k)^- = -(F^{34}_k)^-, \qquad (F^{13}_k)^- = (F^{24}_k)^-, \qquad (F^{14}_k)^- = -(F^{23}_k)^-.$$

Then, using (3.16), we obtain

$$\begin{aligned}(F^+,\ F^-)_V &= tr\sum_k\sum_{i<j}(F^{ij}_k)^+\left[(F^{ij}_k)^-\right]^* = -tr\sum_k\sum_{i<j}(F^{ij}_k)^+\left[(F^{ij}_k)^-\right]^* \\ &= -(F^+,\ F^-)_V.\end{aligned}$$

Thus, $(F^+,\ F^-)_V = 0$. Similarly, we have $(F^-,\ F^+)_V = 0$. □

It should be noted that in the continual case Relation (5.3) implies that the self-dual and anti-self-dual connections (solutions of (2.12)) are always absolute minima of the action S (see [14]).

6 Combinatorial model of the 4-sphere

In this section we discus the question of generalizing our constructions introduced above to the case of a 4-dimensional complex which is the boundary of a 5-dimensional domain. Note that constructions used in [19, 20], namely the operation $*$, are inappropriate to this case. It is convenient to employ here a construction based on the use of the double complex. We will use a standard technique ("gluing of the double") that turns a manifold with boundary into a manifold without boundary.

Let $V \in C(4)$ be a "domain" in the form (3.13). Together with $V \in C(4)$ we introduce its counterpart $\hat{V} \in C(4)$. Considering now $V,\ \hat{V}$ to be two distinct domains and identifying the respective elements of the boundary, we obtain the 4-dimensional combinatorial manifold $M = V \bigcup \hat{V}$ which is homeomorphic to the 4-dimensional sphere. Let $s^{(p)}_k$ be a basis element of $C(V)$. Denote by $\hat{s}^{(p)}_k$ the corresponding basis element of $\hat{C}(V)$. The "gluing" conditions of V and $\hat{V}$ are defined by

$$\begin{aligned} s^{(p)}_{k_1\ldots 0\ldots k_4} &= \hat{s}^{(p)}_{k_1\ldots N_i\ldots k_4}, \qquad & s^{(p)}_{k_1\ldots \tau N_i\ldots k_4} &= \hat{s}^{(p)}_{k_1\ldots 1\ldots k_4}, \\ \hat{s}^{(p)}_{k_1\ldots 0\ldots k_4} &= s^{(p)}_{k_1\ldots N_i\ldots k_4}, \qquad & \hat{s}^{(p)}_{k_1\ldots \tau N_i\ldots k_4} &= s^{(p)}_{k_1\ldots 1\ldots k_4}, \end{aligned} \tag{6.1}$$

where $0 \leq k_i \leq N_i$, see (3.13). On the other hand, a new combinatorial object, namely the complex $C(M)$, is defined by Conditions (6.1). The boundary operator ∂ on $C(M)$ is given by (3.4). We call the complex $C(M)$ a combinatorial 4-dimensional sphere. As in section 3, we introduce the dual complex $K(M)$. No essential modifications are needed to carry out constructions, considered in $K(4)$, in the complex $K(M)$. An arbitrary p-form $\varphi \in K(M)$ can be written as

$$\varphi = \sum_k \sum_{(p)} (\varphi_k^{(p)} s_{(p)}^k + \hat{\varphi}_k^{(p)} \hat{s}_{(p)}^k),$$

where $\varphi_k^{(p)}, \ \hat{\varphi}_k^{(p)} \in gl(2, \mathbb{C})$ and $s_{(p)}^k \in K(V), \ \hat{s}_{(p)}^k \in K(\hat{V})$ are corresponding basis elements, $k = (k_1, k_2, k_3, k_4)$, $k_i = 1, 2, ..., N_i$. Due to the definition (3.13), Conditions (6.1) imply the following conditions for the form φ:

$$\begin{aligned} \varphi^{(p)}_{k_1 \dots 0 \dots k_4} &= \hat{\varphi}^{(p)}_{k_1 \dots N_i \dots k_4}, \qquad & \varphi^{(p)}_{k_1 \dots \tau N_i \dots k_4} &= \hat{\varphi}^{(p)}_{k_1 \dots 1 \dots k_4}, \\ \hat{\varphi}^{(p)}_{k_1 \dots 0 \dots k_4} &= \varphi^{(p)}_{k_1 \dots N_i \dots k_4}, \qquad & \hat{\varphi}^{(p)}_{k_1 \dots \tau N_i \dots k_4} &= \varphi^{(p)}_{k_1 \dots 1 \dots k_4}. \end{aligned} \tag{6.2}$$

Recall that the components $\varphi^{(p)}_{k_1 \dots 0 \dots k_4}$, $\varphi^{(p)}_{k_1 \dots \tau N_i \dots k_4}$ appear when we consider the coboundary operator d^c and its applications.

It is obvious that the double complex construction extends to the complex $C(M)$ (or $K(M)$). The star operation $*$ on $K(M)$ is also defined by (3.13). So, for any p-forms $\varphi, \psi \in K(M)$ the inner product can be written as

$$(\varphi, \ \psi)_M = tr < V_p, \ \varphi \otimes *\psi^* > + tr < \hat{V}_p, \ \hat{\varphi} \otimes *\hat{\psi}^* >$$

or

$$(\varphi, \ \psi)_M = tr \sum_k \sum_{(p)} \left(\varphi_k^{(p)} \left(\psi_k^{(p)} \right)^* + \hat{\varphi}_k^{(p)} \left(\hat{\psi}_k^{(p)} \right)^* \right).$$

The discrete connection 1-form over $C(M)$ is defined by

$$A = \sum_k \sum_{i=1}^4 (A_k^i e_i^k + \hat{A}_k^i \hat{e}_i^k),$$

where $A_k^i, \hat{A}_k^i \in su(2)$, $e_i^k, \hat{e}_i^k$ are 1-dimensional basis elements of $K(M)$, $k = (k_i, k_2, k_3, k_4)$, $k_i = 1, 2, ..., N_i$. Similarly, the discrete curvature 2-form (4.4) can be written as

$$F = \sum_k \sum_{i<j} (F_k^{ij} \varepsilon_{ij}^k + \hat{F}_k^{ij} \hat{\varepsilon}_{ij}^k),$$

where $F_k^{ij}, \hat{F}_k^{ij} \in gl(2, \mathbb{C})$, $\varepsilon_{ij}^k, \hat{\varepsilon}_{ij}^k$ are 2-dimensional basis elements of $K(M)$. The components $A_k^i, \hat{A}_k^i, F_k^{ij}, \hat{F}_k^{ij}$ satisfy Conditions (6.2).

It is easy to check that all constructions from Sections 4, 5 carry out in the complex $K(M)$. Thus, we can write the discrete Yang-Mills equations in the form (4.7), (4.12) and Theorem 4.7 holds on the combinatorial sphere $C(M)$. In $K(M)$ the difference self-dual and anti-self-dual equations (5.2) are completed by the same equations for the components $\hat{F}_k^{ij}$. So, under Conditions 6.2 we obtain the finite-dimensional system of matrices equations.

References

[1] D. Adams, *Global obstructions to gauge-invariance in chiral gauge theory on the lattice*, Nucl. Phys. B **589** (2000), pp. 633-656.

[2] J. Ambjorn, Y. M. Makeenko, J. Nishimura, R. J. Szabo, *Lattice Gauge Fields and Discrete Noncommutative Yang-Mills Theory*, arxiv: hep-th/0004147.

[3] G. A. Baker, *Combinatorial Laplacians and Sullivan-Whitney forms*, in Differential geometry (Proceeding. Special year. Maryland 1981-1982), pp. 1–33, Birkhauser, 1983.

[4] V. de Beaucé, S. Sen, J. C. Sexton, *Chiral dirac fermions on the lattice using geometric discretisation*, Nucl. Phys. B (Proc. Suppl.) **129-130** (2004), pp. 468-470.

[5] D. Birmingham, M. Rakowski, *A star product in lattice dauge theory*, Phys. Lett.B **299** (1993), no. 3-4, pp. 299-304.

[6] L. Castellani, C. Pagani, *Finite Group Discretization of Yang-Mills and Einstein Actions*, Annals of Physics **297** (2002), no. 2, pp. 295-314.

[7] A. A. Dezin, *Multidimensional Analysis and Discrete Models*, CRC Press, Boca Raton, 1995.

[8] A. A. Dezin, *Models generated by the Yang-Mills equations*, Differentsial'nye Uravneniya **29** (1993), no 5, pp. 846–851; English transl.in Differential Equations (1993).

[9] J. Dodziuk, *Finite-difference approach to Hodge theory of harmonic forms*, Amer. J. Math., **98**, 1976, pp. 79–104.

[10] J. Dodziuk, V. K. Patodi, *Riemannian structures and triangulations of manifolds* Journal of Indian Math. Soc. **40** (1976) pp. 1–52.

[11] D. Fried, K. Uhlenbek, *Instantons and Four-Manifolds*, Springer–Verlag, 1984.

[12] S. A. Frolov, *Gauge-invarian Hamiltonian formulation Yang-Mills theory and Heisenberg double*, Mod. Phys. Lett. A **10** (1995), no. 34, pp. 2619-2631.

[13] J. Komorowski, *On finite-dimensional approximations of the exterior differential, codifferential and Laplacian on a Riemannian manifold*, Bull. Acad. Pol. Sci., Vol. **23**, No **9**, 1975, pp. 999–1005.

[14] C. Nash, S. Sen, *Toplogy and Geometry for Physicists*, Acad. Press, London, 1989.

[15] J. Nishimura, *Four-dimensional N = 1 supersymmetric Yang-Mills theory on the lattice without fine-tuning*, Phys. Lett.B **406** (1997), no. 3, pp. 215-218.

[16] E. Seiler, *Gauge theories as a problem of constructive quantum field theory and statistical mechanics*, Lecture Notes in Physics **159** , Springer–Verlag, 1982.

[17] S. Sen, J. C. Sexton, D. Adams, *A geometric discretisation scheme applied to the Abelian Chern-Simons theory*, arxiv: hep-th/0001030.

[18] S. Shabanov, *Infrared Yang-Mills theory as a spin system. A lattice approach*, Phys. Lett.B **522** (2001), no. 1-2, pp. 201-209.

[19] V. Sushch, *Gauge-invariant discrete models of Yang-Mills equations*, Mat. Zametki, **61** (1997), no. 5, pp. 742–754; English transl. in Mathematical Notes, **61** (1997), no. 5, pp. 621–631.

[20] V. Sushch, *Discrete model of Yang-Mills equations in Minkowski space*, Cubo A Math. Journal, **6** (2004), no. 2, pp. 35–50.

[21] V. Sushch, *Discrete models on the 2-sphere*, Dop. NAN Ukrainy, (2000), no 2, pp. 27–32.

[22] H. Whitney, *Geometric integration theory*, Prinston Univ. Press, Prinston, NJ, 1957.

[23] S. O. Wilson, *Geometric structures on the cochains of a manifold*, arxiv: math.GT/0505227.